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DERIVATION AND PROPERTIES OF THE SIZE-BIASED POISSON JANARDAN DISTRIBUTION

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ABSTRACT

In this paper the size-biased Poisson Janardan distribution (SBPJD) is introduced. The probability distribution of size-biased Poisson Janardan distribution is obtained by considering size-biased form of the Poisson distribution and Janardan distribution without its size-biased form. Some of its basic properties are derived and it is found that size-biased Poisson Lindley distribution given by Srivastava and and Adhikari, is a special case of the size-biased Poisson Janardan distribution. The equations of the method of moment and maximum likelihood estimators are obtained to find the estimators of the parameters of the size-biased Poisson Janardan distribution.

KEYWORDS: MLE, MOM, Poisson Distribution, Size-Biased Distribution, SBPJD, SBPLD

1. INTRODUCTION

Shanker (2013) obtained the two parameter Janardan distribution that is the mixture of exponential $\left(\frac{\theta}{\alpha}\right)$ and gamma $\left(2,\frac{\theta}{\alpha}\right)$ distributions with the following probability distribution function (pdf)

$$f(x;\theta,\alpha)=\frac{\theta^2}{\alpha(\theta+\alpha^2)}(1+\alpha x)e^{-\frac{\theta}{\alpha}x}, \qquad x>0, \quad \theta>0,\alpha>0. \tag{1.1}$$

Lindley (1958) introduced a one parameter distribution named Lindley distribution with pdf

$$f(x;\theta)=\frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}, \quad x>0, \quad \theta>0. \tag{1.2}$$

It can be seen that for $\alpha=1$, the Janardan distribution (1.1) is becomes Lindley distribution (1.2).

Shanker et al (2014) obtained the discrete Poisson Janardan distribution (PJD) as

$$f(x;\theta,\alpha)=\left(\frac{\theta}{\theta+\alpha}\right)^2\left(\frac{\alpha}{\theta+\alpha}\right)^x\left(1+\frac{\alpha(1+\alpha x)}{\theta+\alpha^2}\right), \qquad x=0,1,2,....,; \quad \theta>0,\alpha>0. \tag{1.3}$$

For $\alpha=1$, the PJD in (1.3) is becomes the Poisson Lindley distribution (PLD) introduced by Sankaran (1970).

The pdf of size-biased Poisson distribution (SBPD) is

$$P(x;\lambda)=e^{-\lambda}\frac{\lambda^{x-1}}{(x-1)!},\qquad x=1,2,3,....;\quad \lambda>0. \tag{1.4}$$

Adhikari and Srivastava (2013) introduced the size-biased Poisson Lindley distribution (SBPLD) which is obtained by considering the size-biased Poisson distribution with Lindley distribution without considering its size-biased form. Then the pdf of SBPLD

$$f(x;\theta)=\frac{\theta^2}{(1+\theta)^{x+2}}(x+\theta+1),\qquad x=1,2,3.....;\quad \theta>0. \tag{1.5}$$

Adhikari and Srivastava (2014) proposed Poisson size-biased Lindley distribution (PSBLD) considering Poisson without its size-biased version and size-biased version of Lindley. Ghitany and Al-Mutairi (2008) proposed the size-biased Poisson Lindley distribution which is obtained by considering the size-biased version of Poisson Lindley distribution. They discussed the estimation methods for the size-biased Poisson Lindley distribution and its applications on real data sets.

Suppose that the original values of x comes from a distribution with pdf $f_0(x)$ and the values of x recorded according to a probability re-weighted by a weight function $w(x)>0,$ then the pdf of x

$$f(x)=\frac{w(x)}{E(w(x))}f_0(x)$$

Rao (1965) introduced this type of distributions and named them weighted distributions. For $w(x)=x,$ is called size-biased or length-biased distribution. Patil and Rao (1978) showed that the size-biased distributions occur in a usual way in many sampling problems. Patil and Ord (1975) discussed size-biased and related weighted distributions in sampling procedures. Patil and Rao (1977) discussed the applications of size-biased distributions in real-life problems.

2. SIZE-BIASED POISSON JANARDAN DISTRIBUTION (SBPJD)

Suppose that λ of the size biased Poisson distribution in (1.4) follows the Janardan distribution in (1.1). Then the mixture of size-biased Poisson and two parameter Janardan distributions is obtained as

$$f(x;\theta,\alpha)=\int_0^\infty P(x;\lambda)f(\lambda;\theta,\alpha)d\lambda.$$

$$f(x;\theta,\alpha)=\frac{\theta^2}{\alpha(\theta+\alpha^2)}\left(\frac{\alpha}{\theta+\alpha}\right)^x\left(1+\frac{\alpha^2x}{\theta+\alpha}\right),\qquad x=1,2,3,....;\quad \theta>0,\alpha>0. \tag{2.1}$$

The size-biased Poisson Janardan distribution (SBPJD) in (2.1) is reduces to size-biased Poisson Lindley distribution (SBPLD) in (1.5).

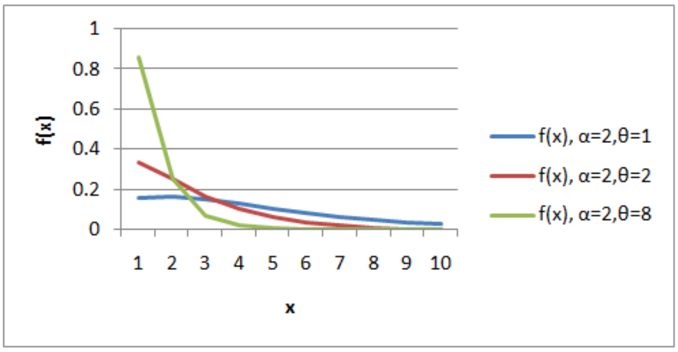

Figure 1: Plot of Pdf SBPLD For $\alpha = 2$ & $\theta = 1,2,8$.

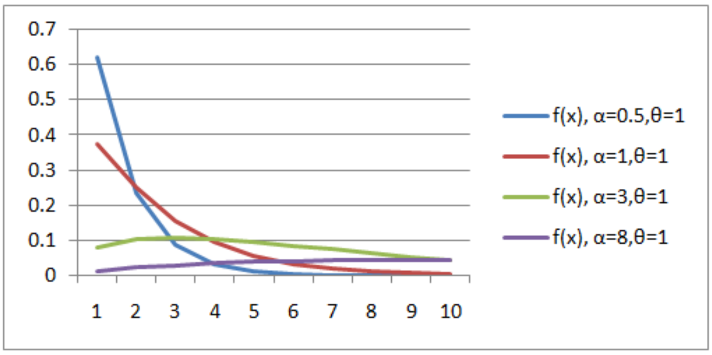

Figure 2: Plot of pdf SBPLD For $\alpha = 0.5,1,3,8$ & $\theta = 1$.

From Figure 1 & 2 it can be seen that the SBPJD is positively skewed. Figure 1 shows that for fix value of $\alpha = 2$ and $\theta = 8$ the pdf has high peaked with longer right tail and for $\theta = 1$ & 2 the pdf is going to be flatter. Figure 2 shows that for fix value of $\theta = 1$ & $\alpha = 0.5$ the pdf is high peaked with longer right tail and for $\alpha = 3$ & 8 the pdf is going to be flatter. Moreover for $\alpha = 1$ the pdf is SBPLD.

Table 1: The Moments of the SBPJD and SBPLD

Measures	SBPJD	SBPLD (For $\alpha = 1$ In SBPJD)
μ_1'	$1+\frac{\alpha\left(\theta+2\alpha^2\right)}{\theta\left(\theta+\alpha^2\right)}$	$\frac{\theta^2+2\theta+2}{\theta(\theta+1)}$
μ_2'	$\frac{\theta^2\left(\theta+\alpha^2\right)+3\theta\alpha\left(\theta+2\alpha^2\right)+2\alpha^2\left(\theta+3\alpha^2\right)}{\theta^2\left(\theta+\alpha^2\right)}$	$\frac{\theta^3+4\theta^2+8\theta+6}{\theta^2(\theta+1)}$
μ_3'	$\frac{\theta^3\left(\theta+\alpha^2\right)+7\theta^2\alpha\left(\theta+2\alpha^2\right)+12\theta\alpha^2\left(\theta+3\alpha^2\right)+6\alpha^3\left(\theta+4\alpha^2\right)}{\theta^3\left(\theta+\alpha^2\right)}$	$\frac{\theta^4+8\theta^3+26\theta^2+42\theta+24}{\theta^3(\theta+1)}$
μ_4'	$\frac{\theta^4\left(\theta+\alpha^2\right)+15\theta^3\alpha\left(\theta+2\alpha^2\right)+50\theta^2\alpha^2\left(\theta+3\alpha^2\right)+60\theta\alpha^3\left(\theta+4\alpha^2\right)+24\alpha^4\left(\theta+5\alpha^2\right)}{\theta^4\left(\theta+\alpha^2\right)}$	$\frac{\theta^5+16\theta^4+80\theta^3+210\theta^2+264\theta+120}{\theta^4(\theta+1)}$
σ^2	$\frac{\theta^3\alpha+\theta^2\alpha^2+3\theta^2\alpha^3+4\theta\alpha^4+2\theta\alpha^5+2\alpha^6}{\theta^2\left(\theta+\alpha^2\right)^2}$	$\frac{\theta^3+4\theta^2+6\theta+2}{\theta^2(\theta+1)^2}$

Table 1 shows the first four moments, mean, variance and of the SBPJD and SBPLD

Some more properties of the size-biased Poisson Janardan distribution are

i. Since

$$\mu-\sigma^2=\frac{\theta^4+2\theta^3\alpha^2+\theta^2\alpha^4-\theta^2\alpha^2-4\theta\alpha^4-2\alpha^6}{\theta^2\left(\theta+\alpha^2\right)^2} \tag{2.2}$$

It follows from (2.2) that $\mu <=> \sigma^2$ for different values of α & θ, that means SBPJD is over-dispersed/equi-dispersed/under-dispersed for different values of α & θ.

Table 2: Dispersion of the SBPJD

$\mu-\sigma^2$, For $\alpha=1$		$\mu-\sigma^2$, For $\theta=1$	
$\theta=0.05$	-798.0930	$\alpha=0$	1
$\theta=0.5$	-6.5556	$\alpha=0.0001$	-0.9999
$\theta=1$	-0.7500	$\alpha=0.001$	-0.9999
$\theta=1.5$	0.4311	$\alpha=0.005$	-0.9999
$\theta=2$	0.6111	$\alpha=0.5$	-0.6
$\theta=8$	0.9811	$\alpha=2$	-1.8
$\theta=10$	0.9883	$\alpha=10$	-2.93
$\theta=15$	0.9950		
$\theta=20$	0.9973		
$\theta=100$	0.9999 $\cong 1$		

From the Table 2 it is observed that for fix value of $\alpha=1$, the SBPJD is over-dispersed as the value of θ is increasing but the SBPJD is under-dispersed as the value of θ is decreasing. For fix value of $\theta=1$, as the value of α increases the SBPJD is under-dispersed and the amount of dispersion increase as well. Therefore it can be seen that for $\theta=100, \alpha=1$ and $\theta=1, \alpha=0$, the SBPJD is qui-dispersed.

ii. Since

$$\frac{f(x+1;\theta,\alpha)}{f(x;\theta,\alpha)}=\frac{\alpha}{\theta+\alpha}\left(1+\frac{\alpha^2}{\theta+\alpha+\alpha^2 x}\right) \tag{2.3}$$

It can be seen that eq(2.3) is a decreasing function in x, ……… $f(x;\theta)$ is log concave. Therefore the SBPJD is unimodal,

The mode of the SBPJD is

$$\text{mod}\, e=-\frac{1}{\ln\left(\frac{\alpha}{\theta+\alpha}\right)}-\frac{\theta+\alpha}{\alpha^2} \tag{2.4}$$

For $\alpha = 1$ it becomes the mode of the SBPLD.

3. METHOD OF MOMENTS (MOM)

Let a random sample of size n from SBPJD with pdf (2.1), the MOM estimates of the θ & α are respectively

$$\tilde{\theta} = \frac{-\left[(\bar{x}-1)\alpha^2 - \alpha\right] + \sqrt{\left[(\bar{x}-1)\alpha^2 - \alpha\right] + 8\alpha^3(\bar{x}-1)}}{2(\bar{x}-1)}, \qquad \bar{x} > 1. \tag{2.5}$$

$$6\alpha^4 + 6\theta\alpha^3 + \theta(2 - \theta b + \theta)\alpha^2 + 3\theta^3\alpha - \theta^3(b-1) = 0 \tag{2.6}$$

Where $b = \dfrac{\sum_{i=1}^{n} x_i^2}{2}$

4. METHOD OF MAXIMUM LIKELIHOOD ESTIMATOR (MLE)

Let a random sample of size n from SBPJD with pdf (2.1), the ML estimates of the θ & α are respectively

$$\frac{2n}{\theta} - \frac{n}{\theta+\alpha^2} - \frac{\sum_{i=1}^{n} x_i}{\theta+\alpha} - \frac{n}{\theta+\alpha} + \sum_{i=1}^{n} \frac{1}{\theta+\alpha+\alpha^2 x_i} = 0 \tag{2.7}$$

$$\frac{n}{\alpha} + \frac{2n\alpha}{\theta+\alpha^2} - \frac{\sum_{i=1}^{n} x_i}{\alpha} + \frac{\sum_{i=1}^{n} x_i}{\theta+\alpha} + \frac{n}{\theta+\alpha} - \sum_{i=1}^{n} \frac{1+2\alpha x_i}{\theta+\alpha+\alpha^2 x_i} = 0 \tag{2.8}$$

The ML estimate of $\hat{\theta}$ and $\hat{\alpha}$ are obtained by the solution of the non-linear equations

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